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ON THE

THEORY OF THE DIVISION OF THE OCTAVE.

The Theory of the Division of the Octave, and the Practical Treatment of the Musical Systems thus obtained. Revised Version of a Paper entitled 'On Just Intonation in Music; with a description of a new Instrument for the easy control of Systems of Tuning other than the equal Temperament of 12 Divisions in the Octave. By R. H. M. Bosanquet, Fellow of St. John's College, Oxford.

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The mode of expressing Intervals.

In the original paper presented by the writer to the Royal Society, logarithms were employed as the measure of intervals, as they have been

commonly employed by others. Great advantages have been found, however, to result from the adoption of the equal temperament (E. T.) semitone, which is $\frac{1}{12}$ of an octave, as the unit of interval. It is the unit most familiar to musicians, and has been found to admit of the expression of the theory of cyclical systems by means of formulæ of the simplest character. The writer therefore devised the following rules for the transformation of ratios into E. T. semitones and *vice versâ*, and subsequently found that De Morgan had given rules for the same purpose which are substantially the same (Camb. Phil. Trans. vol. x. p. 129). The rules obviously depend on the form of log 2. The form of the first rule affords a little more accuracy than De Morgan's.

Rule I. To find the equivalent of a given vibrations-ratio in E. T. semitones.

Take log (ratio), subtract $\frac{1}{300}$, and call this the first improved value. From log (ratio) subtract $\frac{1}{300}$ of the first improved value and $\frac{1}{10,000}$ of the first improved value. Multiply the remainder by 40. We can rely on five places in the result.

The following data are introduced here; they can be verified by numbers given in Woolhouse's tract:—

$$\text{Fifth} = 7{\cdot}019{,}550{,}008{,}654.$$
$$\text{Third} = 4 - {\cdot}136{,}862{,}861{,}351.$$

Five places are ordinarily sufficient.

Rule II. To find the vibrations-ratio of an interval given in E. T. semitones.

To the given number add $\frac{1}{300}$ and $\frac{1}{10,000}$ of itself. Divide by 40. The result is the logarithm of the ratio required. We can rely on five places in the result, or on six, if six are taken.

Ex. The E. T. third is 4 semitones. The vibrations ratio found as above is 1·259921.

Hence the vibrations-ratio of the E. T. third to the perfect third is very nearly 126 : 125.

Definitions.

Regular systems are such that all their notes can be arranged in a continuous series of equal fifths.

Regular cyclical systems are not only regular, but return into the same pitch after a certain number of fifths. Every such system divides the octave into a certain number of equal intervals.

Error is deviation from a perfect interval.

Departure is deviation from an E. T. interval.

Intervals taken upwards are called positive, taken downwards, negative.

Systems are said to be *of the rth order*, positive or negative, when the departure of 12 fifths is $\pm r$ units of the system.

Intervals formed by Fifths.

When successions of fifths are spoken of, it is intended that octaves be disregarded. If the result of a number of fifths is expressed in E. T. semitones, any multiples of 12 (octaves) are cast out. Representing the fifth of any system by $7+\delta$, where δ is the departure of one fifth expressed in E. T. semitones, we form the following intervals amongst others :—

Departure of 12 fifths $=12\delta$

$$(12\times(7+\delta)=84+12\delta, \text{ and } 84 \text{ is cast out}).$$

Two-fifths tone $=2+2\delta$

$$(2\times(7+\delta)=14+2\delta, \text{ and } 12 \text{ is cast out}).$$

Seven-fifths semitone, formed by seven fifths up, $=1+7\delta$

$$(7\times(7+\delta)=49+7\delta, \text{ and } 48 \text{ is cast out}).$$

Five-fifths semitone, formed by five fifths down, $=1-5\delta$

$$(5\times-(7+\delta)=-(35+5\delta), \text{ and } 36 \text{ is added}).$$

The seven-fifths semitone will be denoted by s $(=1+7\delta)$; the five-fifths semitone by f $(=1-5\delta)$.

Regular Systems.

The importance of regular systems arises from the symmetry of the scales which they form.

Theorem α. In any regular system five seven-fifths semitones + seven five-fifths semitones make an exact octave, or $5s+7f=12$.

For the departures (from E. T.) of the 5 seven-fifths semitones are due to 35 fifths up, and those of the 7 five-fifths semitones to 35 fifths down, leaving 12 E. T. semitones, which form an exact octave; or,

$$5(1+7\delta)+7(1-5\delta)=12.$$

Theorem β. **In any regular system the difference between the seven-fifths semitone and the five-fifths semitone is the departure of 12 fifths, having regard to sign; or,**

$$s-f=\text{departure of 12 fifths.}$$

Let δ be the departure of each fifth of the system, then $s=1+7\delta$, $f=1-5\delta$; whence $s-f=12\delta$.

Regular Cyclical Systems.

The importance of regular cyclical systems arises from the infinite freedom of modulation in every direction which is possible in such systems when properly arranged; whereas in non-cyclical systems required modulations are liable to be impossible, owing to the demand for notes lying outside the material provided.

Theorem **i. In a regular cyclical system of the $\pm r$th order the difference between the seven-fifths semitone and five-fifths semitone is $\pm r$ units of the system, or $s-f=\pm r$ units.**

Recalling the definition of rth order ($12\delta=\pm r$ units), the proposition follows from Th. β.

Cor. This proposition, taken with Th. α, enables us to ascertain the number of divisions in the octave in systems of any order, by introducing the consideration that each semitone must consist of an integral number of units. The principal known systems are here enumerated :—

7-fifths semitone $=x$ units.	5-fifths semitone $=y$ units.	Number of units in octave (Th. α) $5x+7y=n$.
	Primary (1st order) *Positive.*	
2	1	17
3	2	29
4	3	41
5	4	53
6	5	65
	Secondary (2nd order) *Positive.*	
11	9	118
	Primary Negative.	
1	2	19
2	3	31
	Secondary Negative.	
3	5	50

Theorem **ii. In any regular cyclical system, if the octave be divided into n equal intervals, and r be the order of the system, the departure of each fifth of the system is $\frac{r}{n}$ E. T. semitones.**

For departure of 12 fifths $=12\delta=r$ units by definition and the unit $=\frac{12}{n}$ E. T. semitones;

$$\therefore \delta=\frac{r}{n}.$$

Theorem **iii. If, in a system of the rth order, the octave be divided int**

n equal intervals, $r+7n$ is a multiple of 12, and $\frac{r+7n}{12}$ is the number of units in the fifth of the system.

Let ϕ be the number of units in the fifth.

Then $\phi \frac{12}{n} = 7 + \delta = 7 + \frac{r}{n}$;

$$\therefore \phi = \frac{7n+r}{12},$$

and ϕ is an integer by hypothesis; whence the proposition.

Cor. From this proposition we can deduce corresponding values of n and r. It is useful in the investigation of systems of the higher orders. Casting out multiples of 12, where necessary, from n and r, we have the following relations between the remainders:—

Remainder of

n	1	2	3	4	5	6	7	8	9	10	11
r	5	10	3	8	1	6	11	4	9	2	7

Theorem iv. If a system divide the octave into n equal intervals, the total departure of all the n fifths of the system $= r$ E. T. semitones, where r is the order of the system.

For by Th. ii. $\delta = \frac{r}{n}$; whence

$$n\delta = r,$$

or the departure of n fifths $= r$ semitones.

This gives rise to a curious mode of deriving the different systems.

Suppose the notes of an E. T. series arranged in order of fifths, and proceeding onwards indefinitely, thus:—

c g d a e b f♯ c♯ g♯ d♯ a♯ f c g . . . ,

and so on. Let a regular system of fifths start from *c*. If they are positive, then at each step the pitch rises further from E. T. It can only return to *c* by sharpening an E. T. note.

Suppose that *b* is sharpened one E. T. semitone, so as to become *c*; then the return may be effected

at the first *b* in 5 fifths,
at the second *b* in 17 fifths,
at the third *b* in 29 fifths; and so on.

Thus we obtain the primary positive systems. Secondary positive systems may be got by sharpening *b*♭ 2 semitones; and so on.

If the fifths are negative, the return may be effected by depressing *c*♯ a semitone in 7, 19, 31 . . . fifths; we thus obtain the primary negative systems; or by depressing *d* two semitones, by which we get the secondary negative systems; and so on.

An instructive illustration may be made as follows; it requires too large dimensions for convenient reproduction here:—

Set off on the axis of abscissæ the equal temperament series in order of fifths, as above, taking about 10 complete periods. If the distances of the single terms are made 1 centimetre, this will take 1m·20 in length, starting from the origin on the left.

Select a unit for the E. T. semitone of departure, say 1 decimetre.

Rule a series of lines parallel to the axis of abscissæ, at distances representing integral numbers of E. T. semitones, both above and below.

Rule, parallel to the axis of ordinates, straight lines through the points representing the E. T. notes.

Enter on the intersections the names of the E. T. notes they represent. Thus the notes on the positive ordinate of *c* are *c c*♯ *d* . . . , and so on, each pair separated by 1 decimetre, and the notes on the negative ordinate of *c* are *c*–*b*–*b*♭ . . . .

If we then join the *c* on the left hand of the axis of abscissæ to all the other *c*'s on the figure, except, of course, those on the axis, we obtain a complete graphic representation of all the systems whose orders are included. The rth order is represented by lines drawn to the *c*'s in the rth line above, the −rth by the lines drawn to the *c*'s in the rth line below.

This illustration brings specially into prominence the singularity of multiple systems, as all the multiples of any system lie on the same straight line with it, and the representation fails to give all the notes of such systems.

Multiple Systems.

Multiple systems are such that the number of divisions in the octave (kn) in any such system is a multiple (k) of the number of divisions (n) of some other system.

Multiple systems have not been as yet practically applied.

These systems are not strictly regular; for though their fifths are all equal, yet they do not form one continuous series, but several. They are strictly cyclical, *i. e.* they divide the octave into n equal intervals.

Theorem v. A multiple system, kn, may be regarded as being of order kr, where n is a system of order r.

For, n being a system of order r, $r+7n$ is a multiple of 12; $\therefore$ also $k(r+7n)$ is a multiple of 12, which is the condition that the system kn be of order kr.

This is useful in the investigation of systems of the higher orders.

If n is a multiple of 12, the system is a multiple of the E. T., and of order zero.

In the illustration described under Th. iv. the notes of a multiple system (kn) are the same as those of system n, until the latter is complete. The rest of the representation consists simply of the same notes repeated over and over again. To obtain the rest of the notes we should have to change the starting-point.

On the whole, we may regard the system kn as consisting of k different systems n, having starting-points distant from each other by $\frac{1}{k}$ of the unit of the system n.

It follows immediately that the system kn is of the krth order; for in every unit of the system n there are k units of system kn; and so in r units of system n there are kr units of system kn.

Any system, when n is not a prime, can be regarded as a multiple system.

Thus the system of 59 is of the 7th order; 118 consequently a multiple system of the 14th order, in which point of view it is of no interest; but, casting out the 12 from the order, it may be also regarded as an independent system of the 2nd order, in which point of view it is of considerable interest.

Formation of Major Thirds in Positive and Negative Systems.

The departure of the perfect third is $-\cdot 13686$. Hence negative systems (where the fifth is $7-\delta$) form their thirds in accordance with the ordinary notation of music. For if we take 4 negative fifths up, we have a third with negative departure (-4δ) which can approximately represent the departure of the perfect third. Thus *c*♯ is either the third to *a*, or four fifths up from *a*, in accordance with the usage of musicians.

Positive systems form their thirds by 8 fifths down; for their fifths are of the form $(7+\delta)$, and 8 fifths down give the negative departure (-8δ). Thus the third of *a* should be *d*♭, which is inconsistent with musical usage. Hence positive systems require a separate notation. Helmholtz proposed a notation for this purpose, which, however, is unsuitable for use with written music. The following notation is here adopted for positive systems in general; it is not intended to be limited to any one system, like Helmholtz's. In fact it may, on occasions, be used even for negative systems.

Notation for Positive Regular Systems.

The notes are arranged in series, each containing 12 fifths, from *f*♯ up to *b*. These may be called duodenes, adopting a term introduced by Mr. Ellis. The duodene

f♯–*c*♯–*g*♯–*d*♯–*a*♯–*f*–*c*–*g*–*d*–*a*–*e*–*b*,

which contains the standard *c*, is called the unmarked duodene. No distinction is made in these series between such notes as *c*♯ and *d*♭. These signs refer only to the E. T. note from which the note in question is derived; the place in the series of fifths is determined by the notation.

Continuing the series to the right, each note of the next 12 fifths is affected with the mark / (mark of elevation), drawn upwards in the direction of writing. These notes join on to the unmarked duodene as follows:—

e–*b*–/*f*♯–/*c*♯–/*g*♯ . . . .,

and so on.

Thus /*c* is 12 fifths to the right of *c*, and the interval /*c*–*c* is the departure of 12 fifths.

The next duodene to the right is affected with the mark //, which joins on to the last as before:—

/*e*–/*b*–//*f*♯ . . . .,

and so on.

Proceeding in the same way, we have notes affected with such marks as ///, ////.

Return to the unmarked duodene, and let it be continued to the left; the notes in the next duodene on the left are affected with the mark \, (mark of depression), drawn downwards in the direction of writing. The junction with the unmarked duodene will be

c–*g*–*d*–*a*–*e*–*b*–*f*♯–*c*♯ . . . .

The next junction on the left will be

e–*b*–*f*♯ . . . .;

and, proceeding in the same way, we have such marks as \\\, \\\\.

Thus *c*–*e* is a major third determined by eight fifths down in the whole series; and *e* will have the departure (-8δ) from the E. T. note *e* derived from *c*.

Notation applicable to all Regular Systems, Negative as well as Positive.

As this notation simply consists of a determination of position in a continuous series of fifths, it may be applied to all regular systems, positive or negative; but, as it is not commonly needed for negative systems, it is not generally applied to them.

Formation of Harmonic Sevenths in Positive and Negative Systems.

The harmonic seventh is the interval whose ratio is 7 : 4. It affords a smooth combination, free from beats.

The departure of the harmonic seventh from the note which gives the E. T. minor seventh is $-\cdot 31174$ (Rule I.).

Helmholtz observes that his system of just intonation affords an approximation to the harmonic seventh. In fact, if we form a seventh by 14 fifths down in positive systems (fifth $=7+\delta$), we obtain a note with negative departure (-14δ), which can approximately represent the harmonic seventh: $c-\flat b\flat$ represents such an interval.

Mr. Ellis has observed (Roy. Soc. Proc. 1864) that the mean-tone system, which is negative, affords a good approximation to the harmonic seventh. In fact, if we form a seventh by 10 fifths up in negative systems (fifth $=7-\delta$), we obtain a note with negative departure (-10δ), which can approximately represent the harmonic seventh.

Concords of Regular and Regular Cyclical Systems.

These considerations permit us to calculate the departures and errors of concords in the various regular and regular cyclical systems. There is, however, one quantity which may be also conveniently taken into consideration in all cases, viz. the departure of 12 fifths of the system. We will call this Δ, putting $\Delta=12\delta$.

We have then the following Table of the characteristic quantities for the more important systems hitherto known.

The value of the ordinary comma $\left(\frac{81}{80}\right)$ is ·21560. It is comparable with the values of Δ, and if introduced in its place in the Table would give rise to a regular non-cyclical system, lying between the system of 53 and the positive system of perfect thirds, the condition of which would be that the departure of 12 fifths = a comma.

Name, or n.	Order, r.	$\Delta=12\delta$, or $12\frac{r}{n}$.	Error of fifth, $\delta-\cdot01955$.	Error of third, $\cdot13686-8\delta$.	Error of harmonic seventh, $\cdot31174-14\delta$.
17	1	·70588	·03927	−·33373	−·51178
29	1	·41379	·01493	·27586	−·17101
41	1	·20268	·00484	−·19512	−·02970
Perfect fifths.	...	·23460	...	−·01954	·03804
53	1	·22642	−·00068	−·01409	·04758
Positive perfect thirds.		·20529	·00244	...	·07223
118	2	·20339	−·00260	·00127	·07445
65	1	·18462	−·00417	·01378	·09635
($\delta=\frac{r}{n}$ is here negative.)				$\cdot13686+4\delta$.	$\cdot31174+10\delta$.
43	−1	−·29707	−·04431	·03784	·06418
31	−1	−·38710	−·05181	·00783	−·01084
Mesotonic. Negative perfect thirds.	...	−·41058	−·05376	...	−·03041
50	−2	−·48000	−·05955	−·02314	−·08826
19	−1	−·63158	−·07218	−·05367	−·21458

A few systems of the higher orders, which possess some interest, will be given separately.

An illustration may be made as follows, which shows on inspection all the data involved in the above Table, and the properties of any other system introduced into it.

Take axes of abscissæ and ordinates, and set off on both distances representing tenths of E. T. semitones—for ordinary purposes 10 inches to the E. T. semitone answers best; for Lecture scale, 1 metre to the E. T. semitone.

On the axis of ordinates set off points representing the values in column Δ of the Table, and corresponding values for any other system required. Through each of these points rule a straight line parallel to the axis of abscissæ.

On the axis of abscissæ set off points representing the values −·13686 and −·31174. Rule lines through these parallel to the axis of ordinates. These abscissæ represent respectively perfect thirds and perfect sevenths.

Draw lines inclined to the axis of abscissæ at angles $\tan^{-1}\frac{3}{2}$ and $\tan^{-1}\frac{6}{7}$. These give, by their intersections with the lines of the different positive systems, the thirds and sevenths respectively.

Draw lines inclined to the axis of abscissæ at angles $\tan^{-1}-3$ and $\tan^{-1}-\frac{6}{5}$. These give, by their intersections with the lines of the different negative systems, the thirds and sevenths respectively.

The errors of the thirds and sevenths are the perpendicular distances of the intersections which determine them from the ordinates of perfect thirds and sevenths already constructed.

In Regular Cyclical Systems, to find the number of Units in any Interval in the Scale.

Let x be the number of units in the seven-fifths semitone, then

$$x\cdot\frac{12}{n}=1+7\delta=1+7\frac{r}{n},$$

or

$$x=\frac{n+7r}{12}.$$

It is easy to see that x will always be integral if the order condition is satisfied (Th. iii.), viz. if $7n+r$ is a multiple of 12.

For then $7(7n+r)=49n+7r$; whence, casting out $48n$, $n+7r$ is a multiple of 12.

We can now determine the remaining intervals in terms of x and r:—

Interval.	No. of units. Positive systems.		Negative systems.
5-fifths semitone	$x-r$	......	$x-r$
Minor tone / 10-fifths tone	$2x-2r$	......	..
Major tone / 2-fifths tone	$2x-r$	......	$2x-r$
Minor third	$3x-r$	......	$3x-2r$
Major third	$4x-3r$	......	$4x-2r$
Fourth	$5x-3r$	......	$5x-3r$
Fifth	$7x-4r$	......	$7x-4r$
Sixth	$9x-6r$	......	$9x-5r$
Harmonic seventh	$10x-7r$	......	$10x-5r$
Major seventh	$11x-7r$	......	$11x-6r$
Octave	$12x-7r$	......	$12x-7r=n$.

The $-r$'s in negative systems are, of course, positive quantities.

Employment of Positive Systems in Music.

Rule for thirds.—If we write down one of the duodenes of the notation,

f♯–c♯–g♯–d♯–a♯–f–c–g–d–a–e–b,

and remember that positive systems form their thirds by 8 fifths down, we have the rule:—

The four accidentals on the left in any duodene of the notation form major thirds to the four notes on the extreme right in the same duodene. All other notes have their major thirds in the next duodene below. Thus *d–f♯*, *e–⟍e* are major thirds.

Use of the Notation with Musical Symbols.

It is an essential point in this notation that it can be used with musical symbols. The following example shows the major and minor chords and the interval used for the harmonic seventh:—

The first chord is the major triad; the second involves *g–⟍f*, the harmonic seventh; the fourth crotchet gives the minor common chord; and the first chord of the second bar is the sharp sixth, rendered peculiarly smooth by employment of the approximate harmonic seventh for the interval *⟋a♭–f♯*.

The employment of positive systems is presupposed with this notation, unless the contrary is expressly stated.

Such passages as this can be played on the harmonium hereafter described.

Principle of Symmetrical Arrangement in Regular Systems.

If we place the E. T. notes in the order of the scale, and set off the departures of the notes of any regular system at right angles to the E. T. line, sharp departures up and flat departures down, we obtain the positions of what may be called a symmetrical arrangement.

The distances of the E. T. notes from the starting-point are abscissæ, and the departures ordinates.

Positive Systems.

The subjoined is a symmetrical arrangement of the notes of General Thompson's enharmonic organ (p. 402). It is selected as not being too extensive for reproduction, as being of historical interest, and as illustrating the nature of the difficulty caused by the distribution of such systems into separate key-boards. Each of the single vertical steps represents the departure of one fifth.

The property of symmetrical arrangements, from which they derive their principal importance, is that, position being determined only by relations of interval, the notes of a combination forming given intervals present always the same form, whatever be the key or the actual notes employed.

Let us express, as before, the number of E. T. semitones, which is now our abscissa, by simple integers, and the number of departures of fifths, which is our ordinate, by a coefficient attached to δ. Then we have only to note the values of the different intervals to obtain their coordinates with respect to any note taken as origin.

Thus the third is $4-8\delta$, or four steps to the right and eight down (*c–⟍e*); the fifth is $7+\delta$, seven steps to the right and one up (*c–g*); the minor third is three to the right and nine up (*⟍e–g*); and so on.

Two notes are omitted from the otherwise complete series, *b* and *⟍⟍d*; and we notice the number of otherwise complete chords which their absence destroys.

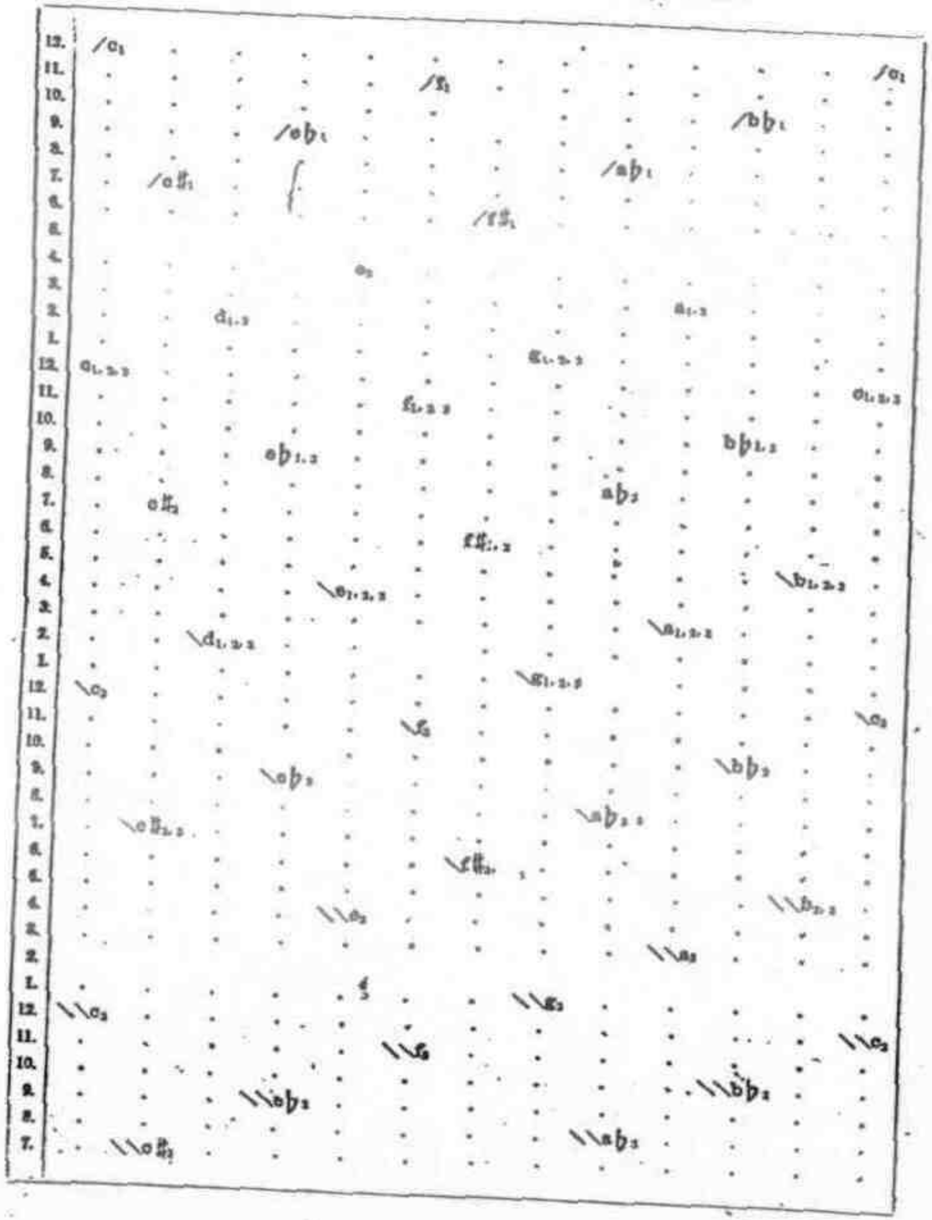
Distribution over three Key-boards.—As an example of the effect of this, we note that the notes of the chord of *a minor* are all present; but they are $a_{1.2}$–$/c_1$–e_2, so that the third and fifth are on different key-boards.

Negative Systems.

According to the enunciation of the principle of symmetrical arrangement, the positions should be taken lower for negative systems as we ascend in the series of fifths; but it is practically more convenient to use the positive form in negative systems as well. The coordinates of some intervals become different—the third is $4+4\delta$, the minor third $3-3\delta$, &c.

Symmetrical Arrangement of the Notes of Thompson's Enharmonic Organ.

The subscripts 1, 2, 3 refer to its three key-boards.



Application of Principle of Symmetrical Arrangement to a "Generalized Key-board" for Regular Systems.

A key-board has been constructed, on the principle of "symmetrical arrangement," in the following manner :—

The octave is taken $=6$ inches horizontally (in ordinary key-boards the octave is $6\frac{1}{2}$ inches). This is divided into 12 spaces, each $\frac{1}{2}$ inch broad. These are called the 12 principal divisions of the octave. A horizontal line gives the positions of an E. T. series where it crosses them all.

The keys are then placed at vertical and horizontal distances from the E. T. line corresponding to their departures, on the supposition that the arrangement is positive.

The departure of 12 fifths up corresponds to a horizontal displacement of 3 inches from the player, and a vertical displacement of 1 inch up.

These displacements are divided equally among the fifths to which they may be regarded as due, *i. e.* the displacement of *g* with respect to *c* is $\frac{1}{4}$ inch back and $\frac{1}{12}$ inch up; so of *d* with respect to *g*, of *a* with respect to *d*, and so on.

Although only 3 inches of each key are thus exposed on a plan, yet the keys are all made to overhang $\frac{1}{2}$ inch, and thus the tangible length of each key is $3\frac{1}{2}$ inches.

The accompanying figure (p. 404) shows a small portion of the key-board, on a scale of half the real size.

The keys are each $\frac{3}{8}$ inch broad, and their centres are $\frac{1}{2}$ inch apart. There is thus $\frac{1}{8}$ inch free between the adjacent surfaces of each pair of keys, and $\frac{5}{8}$ inch altogether between the two keys which rise on each side of any given key. This is of importance; *e. g.*, in the chord *c*-╲*e*-*g*-*c*, taken with the right hand, the first finger has to reach ╲*e* between *d*♭-*f* and under the overhanging *e*.

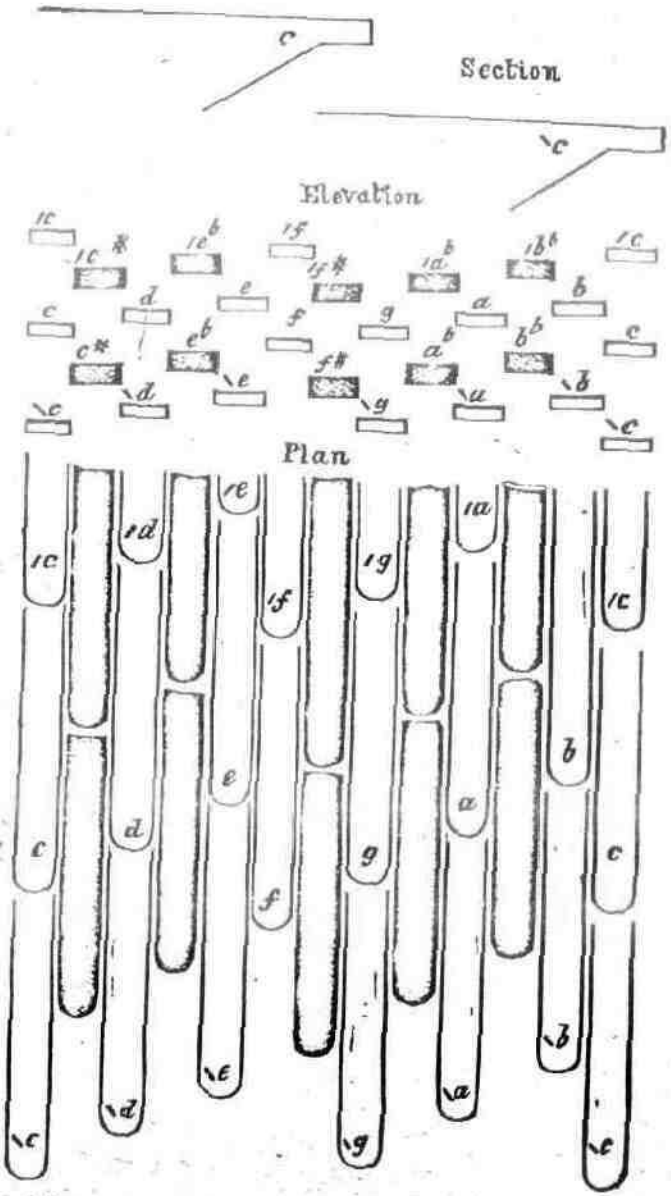
The keys in the five principal divisions which have "accidental" names (e. g. *c*♯ or *d*♭) are black, the rest white.

There are seven keys in each principal division; the seven *c*'s are marked from ╲╲╲*c* to ///*c*, the unmarked *c* being in the middle. Thus there are 84 keys in each octave. The key-board controls an harmonium which contains the system of 53.

Application of the Positive System of Perfect Thirds to the "Generalized Key-board" (Helmholtz's system, just intonation).

If the thirds, such as *c*-╲*e*, are made perfect, and the fifths flat by ·00244, a quantity which escapes the ear, we have the system here mentioned. Helmholtz makes a mistake in describing it ('Die Lehre von den Tonempfindungen,' ed. 3, p. 495); he supposes that the fifths are sharp instead of flat by the above interval; it is easy to see from the context that this is a mistake.

The notation of positive systems is applicable without specialization.



Application of the Notation of Positive Systems to the System of 53.

The notation introduced for positive systems is susceptible of various accessory rules, according to the system it is attached to. In the harmonium to which the above-mentioned key-board belongs the system

of 53 is adopted. It is required to find rules of identification for passing from one principal division of the octave to another.

Rule.—In the system of 53 the notation *of positive systems* becomes subject to the following identifications:—

If two notes in adjoining principal divisions (e.g. *c* and *c*♯) be so situated as to admit of identification (*e.g.* a high *c* and a low *c*♯), *they* will be the same if the sum of the elevation- and depression-marks =4; unless *the lower of* the two divisions is black (accidental), then the sum of the marks of identical notes =5.

This can only be proved by enumeration of a case in each pair of divisions. This enumeration is *made* in the writer's original paper. It is founded on the following principles:—

Noting that the 5-fifths semitone is 4 units (scheme following Th. i.), we see that *c*–*c*♯ is 4 units, whence ////*c*–*c*♯, ///*c*–*c*♯, //*c*–*c*♯ . . . . are identities; or, again, *c*♯–*d* is 4 units, and ////*c*♯–*d*, ///*c*–*d* . . . . are identities.

Application of the System of 53 *to the "Generalized Key-board."*

An harmonium has been constructed *which is arranged* as follows:—

The note *c* is taken as the first note of the series, and receives the characteristic number 1. Then *c* is 4, and the remaining numbers can be assigned by the rules for the identifications in the system of 53 *given* above.

A *number* of notes at the top of the key-board are thus identical with corresponding notes in the adjacent principal divisions on the right at the bottom, *e.g.* //*c*=*b*=*c*♯. These permit the infinite freedom of modulation which *is the characteristic* of cyclical systems; for in moving upwards on the key-board we can, on arriving near the top, change the hands on to identical notes near the bottom, and so proceed further in the same direction, and *vice versâ.*

It is to be noted that, in positive systems, displacement upwards or downwards on the key-board takes place most readily by modulation between related major and minor keys—not, as has been *commonly* assumed, only by modulation round the circles of fifths. In negative *systems,* on the contrary, displacements take place only by modulations of the latter type.

Application of the System of 118 *to the "Generalized Key-board."*

The 5-fifths semitone is here 9 units, and the 7-fifths semitone is 11 units. The major tone (2-fifths tone) is consequently 20, and the minor tone (10-fifths tone) is 18. Hence the notes in the successive principal divisions are alternately odd and even, and the identifications lie in alternate columns. *These are not* here further investigated, as no practical use has been made of the system.

If $c=1$, c♯ $=10$, /c♯ $=12$, $d=21$, ...

It would be possible to construct a key-board on the principles already explained, which would give complete control over the notes of the system of 118. A portion of such a key-board would be practically indistinguishable from one tuned to the positive system of perfect thirds, as the error of the thirds of the system of 118 is too small to be perceived by the ear.

Application of the Negative System of Perfect Thirds (Mean-Tone System) to the "Generalized Key-board."

If the thirds, such as *c–e*, are made perfect, and the fifths ·05376 flat, we have the mean-tone system. The forms of scales and chords in negative systems are different from those in positive systems. The scales are very easy to play, and the chords also. It is expected that this application may prove of practical importance.

Following the scale of unmarked naturals on the plan, we can realize the nature of the fingering. It is the same as that of the Pythagorean scale with the system of perfect fifths. The tones are all 2-fifths tones, and the semitones both 5-fifths semitones.

Application of the Negative System of 31 *to the "Generalized Key-board."*

The fifths are a little better than in the last case, viz. ·05181 flat; the thirds ·00783 sharp. The only difference in the employment of the system is that the arrangement is cyclical. The tones all consist of five units, semitones of three.

The Investigation of Cycles of the Higher Orders—the new Cycle of 643 *and others.*

The system of 301 is of interest, as combining the properties of a tolerably good positive cyclical system with the representation of intervals accurately to three places by means of logarithms. This system has been lately used, in particular by Mr. Ellis, for approximate calculations. It appears to be of some interest to investigate generally what systems of higher orders do represent either of the systems with perfect thirds, and with what degree of accuracy they do so.

First, with respect to *positive systems*. If a system n of the rth order be a close approximation to the system of perfect thirds, then will $-8\frac{r}{n}$ (the departure of its third) approximate in value to $-·13686$; or

$$\frac{r}{n}=\frac{·13686}{8}=\frac{1}{58·4526} \text{ nearly,}$$

or

$$n=r\ 58·4526 \text{ nearly.}$$

Now, when $r=2$ we have the system of 118, which affords the closest approximation to what is required of any cyclical system known hitherto, the error of its third being ·00127.

Referring to Th. v., it is easy to see that no other even system of an order much below the 24th can afford a better approximation; for the number 118 differs from the value given by the above condition by little more than unity. Its multiple is always of the right order (Th. v.); there can therefore be no other system of the right order within 12 digits of the multiple either way, and the deviation of the value given by the condition cannot amount to 12 digits till near the 24th order; we therefore confine ourselves to systems of uneven orders.

Casting out 12's from 58·4526, we can take the remainder as 10·45 for the purposes of the search:—

r.	r. 10. 45.	Remainder, casting out 12's.	Remainder required for order r (Th. iii.).
3	31·35	7·35	3
5	52·25	4·25	1
7	73·15	1·15	11
9	94·05	10·05	9
11	114·95	6·95	7

The coincidence at the 11th order is the closest so far; and it is easy to see, by considerations analogous to those above, that no subsequent system can afford another till a much higher order is reached.

For the 11th order, then, we have

$$11\times 58·4526=642·9786;$$

and 643 is a system of the 11th order, as shown by its giving remainder 7 on dividing by 12 (Th. iii.).

Calculating the third of this system $\left(\frac{8.11}{643}=\text{dep.}\right)$, and taking seven places, we have:—

$$\begin{aligned}\text{Departure of perfect third} &= -·1368629\\ \text{Departure of third of 643} &= -·1368585\\ \text{Error} &= ·0000044 \text{ sharp.}\end{aligned}$$

To five places both thirds are represented by $-·13686$.

The intervals of this system will furnish us with simple numerical ratios, which represent with great accuracy the intervals of the perfect system.

We have (see the section on the number of units in any interval)—

7-fifths semitone = 60 units,
5-fifths semitone = 49 units;

whence we can deduce the remaining intervals. These values of the semitones suggest the following curious derivation of this system :—

Referring to the Table of characteristic numbers, we notice that the errors of the thirds of the systems of 53 and 65 are nearly equal and opposite.

The system of 53 is derived on the assumption that the interval ratio of the semitones is $\frac{4}{5}$ (Th. i. Cor.), and that of 65 on the assumption $\frac{5}{6}$ for the same ratio; taking, then, an intermediate ratio, $\frac{9}{11}$, we get the system of 118, which has very good thirds.

But if we take an intermediate ratio in the following manner, we get the new system of 643 :—

Reducing the fractions $\frac{4}{5}$, $\frac{5}{6}$ to a common denominator, we have $\frac{24}{30}$, $\frac{25}{30}$, or doubling, $\frac{48}{60}$, $\frac{50}{60}$; and if we take the intermediate ratio $\frac{49}{60}$, we get the system of 643, by the formula $5x+7y=n$, derived from Th. a of Regular Systems.

The systems of the fifth order are not particularly good; the best is 289, then 301. They derive their interest from the logarithmic properties of 301.

Negative Systems.—The condition for the excellence of the thirds of negative systems is that

$$4\frac{r}{n} = -\cdot 13686 \text{ nearly,}$$

or

$$\frac{r}{n} = \quad 29\cdot 2263 \text{ nearly.}$$

Searching as before, we find for order -7,

$$7\times 29\cdot 2263 = 204\cdot 5841;$$

and 205 is a system of order -7.

Comparing thirds,

Departure of perfect third = $-\cdot 1368629$
Departure of third of 205 = $-\cdot 1369002$

Error = $\cdot 0000373$ flat.

The following is a *résumé* of the properties of these higher systems :—

System.	Order.	$\Delta=12\delta$.	Error of fifth.	Error of third.
289	5	·20761	−·00225	−·00155
643	11	·20529	−·00244	+·0000044
301	5	·19934	−·00294	+·00397
205	−7	−·41070	−·05377	−·000037