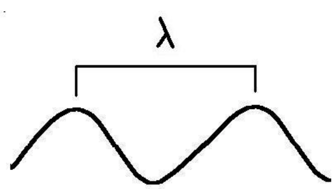
CHAPTER 14: MECHANICAL WAVES

Objectives: In this chapter, we'll learn about the different kinds of mechanical waves, especially transverse waves, like on a vibrating string, and longitudinal waves, like sound waves. We'll find out about standing waves, about the nodes and antinodes of a wave, the frequency, wavelength, and velocity of waves. We'll also study constructive and destructive interference, beats, and the doppler shift of sound waves.

14.1 Introduction

In **transverse waves**, the wave motion is perpendicular to the direction propagation, such as the vibrations of a guitar string. **Longitudinal waves** have wave motion in the same direction as the direction of propagation, such as when a slinky is slapped horizontally on the end. Sound waves are longitudinal. Ocean waves are a combination of the two. Waves rarely appear in isolation, and when they do they're called **Solitons**. Some scientists think elementary particles might best be described by soliton waves. Waves are usually created in continuous media, where the energy and momentum is transferred from particle to particle, so that the particles do not move individually very far, other than to oscillate around a point of equilibrium as the energy and momentum of the wave passes through. Waves are represented mathematically by wave functions. The most common wave functions are sines and cosines.



There are several important physical quantities that must be defined. The ***wavelength***, λ, is the distance the wave travels before repeating itself (see the figure). The ***frequency***, f, is the number of complete cycles that a wave goes through every second. A point on a vibrating string, for example, will start off at a midpoint, then go up to its maximum, down to a minimum, and back to the start. This corresponds to one cycle. It may also be thought of as the passage of one wave in a train of waves. The number of times this happens in a second is the frequency. **Frequency** is measured in **cycles per second**, or **Hertz**. The propagation speed v of the wave is such that

$$v = \lambda f$$

Example 1. A dude on a dock starts timing wavecrests, starting his watch when the first crest reaches a pylon. Five seconds later, a third crest is just reaching the pylon. He also estimates the distance between crests is four meters (A) What is the wavelength? (B) the frequency? (C) the speed of the wave?

Solution: (A) The wavelength is four meters, which is the distance from crest to crest. (B) To get the frequency, you need to find the number of waves that pass in a given time. Though three crests are involved, only two complete waves passed in five seconds–the third wave had just started at the end of five seconds. So

$$f = \frac{2\ waves}{5\ seconds} = 0.4\ \frac{waves}{second} = 0.4\ Hertz$$

(C) To get the speed, just multiply the wavelength and frequency.

$$v = f\lambda = 0.4 \cdot 4 = 1.6 \quad m/s$$

Other useful quantities include the **angular frequency**, ω, the number of radians per second, and K, the **wave number**, which gives the number of waves per meter. In addition, there is the **angular wave number**, $\boldsymbol{k}$, which is the number of radians per meter. The reason radians, a measure of angle, is involved, is that wave motion can be related to motion in a circle, since it repeats. There are some easy relationships between these ideas.

$$\omega = 2\pi f$$

while

$$\mathrm{K} = \frac{1}{\lambda}$$

and

$$k = 2\pi \mathrm{K}$$

A typical transverse wave propagates in the x-direction while flapping up and down in the y-direction. The amplitude, $\boldsymbol{A}$, is the maximum displacement of the wave in the $\boldsymbol{y}$-direction. A sinusoidal solution has the form

$$y = A \sin\left(\omega t - kx + \phi\right)$$

where ϕ is a phase shift. Since the wave equation is linear, different such solutions can be added together, and the sum is still a solution. This is called the **Principle of Superposition**. Cosine functions (and many others) can also be used.

Example 2. Let a wave function be given by $\psi = 20 \sin(5t - 12x)$. Find the amplitude, the angular frequency, the frequency, the angular wave number, the wave number, the wavelength, and the velocity of propagation of the wave.

Solution: The amplitude is just the number in front of the sine: ***A=20 meters***. The angular frequency and angular wave number can be read off from the numbers in front of ***t*** and ***x***, respectively:

$$\omega = 5 \ rad/s \qquad\qquad k = 12 \ \ rad/meter$$

The rest of the quantities can be obtained from these two.

$$f = \frac{\omega}{2\pi} = 0.796 \ hz \qquad\qquad \mathrm{K} = \frac{k}{2\pi} = 1.91 \ m^{-1}$$

$$\lambda = \frac{1}{\mathrm{K}} = 0.524 \ m. \qquad\qquad c = f\lambda = 0.417 \ m/s$$

14.2 Waves on a String

With a lot of effort it can be found that the velocity **v** of a wave on a string is given by :

$$v = \sqrt{\frac{T}{\mu}}$$

where **T** is the tension in the string. We also have $\mu = m/L$, linear density of the string, in kilograms per meter, ***m*** being the mass and ***L*** the length of the string.

Example 3. A steel wire is stretched to a tension of 500 Newtons. If the mass is 40 gm and the length is 2 meters, what is the velocity of waves on the wire?

Solution: The linear density is

$$\mu = \frac{m}{L} = \frac{0.04\ kg}{2\ m} = 0.02\ \frac{kg}{m}$$

The tension in the wire is given, so

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{500}{0.04}} = 111.8\ m/s$$

Example 4. Suppose waves traveling on a wire have an amplitude of 0.02 m, while the tension in the wire is 400 Newtons, and the mass 3 grams/ meter. What is the wave function, assuming it is sinusoidal and the wavelength is two meters? Assume also that at t=x=0, y=0.02 m.

Solution: This is a matter of assembling all the pieces. We have

$$y = A \sin\left(\omega t - kx + \phi\right)$$

A=0.02 m. Next we have to find the angular frequency and angular wave number, and phase shift. We need the wavelength. Since the wire vibrates at the fundamental frequency, the length of the wire is half the wavelength, so $\lambda = 2\ meters$. The velocity can be found with the standard equation,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{300}{0.003}} = 316.2\ \ m/s$$

and then the frequency is

$$f = \frac{v}{2L} = \frac{316.2}{2} = 158.1 \ hz$$

The angular frequency and angular wave number are, therefore,

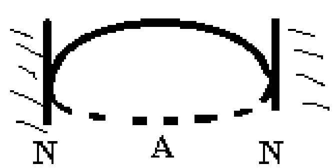
$$\omega = 2\pi f = 2\pi \cdot 158.1 = 316\pi \ \frac{rad}{s} \qquad k = \frac{2\pi}{\lambda} = \frac{2\pi}{2} = \pi \ \frac{rad}{m}$$

When x=t=0, we are told that y=0.02 m. So

$$0.02 = 0.02 \sin\phi \Rightarrow \sin\phi = 1 \Rightarrow \phi = \frac{\pi}{2}$$

The full mathematical expression for this wave is therefore

$$y = 0.02 \sin\left(316\pi t - \pi x + \frac{\pi}{2} \right)$$



Standing Waves on Strings. A standing wave is one in which the points of maximum and minimum displacement don't change with time. The wave appears to be "standing still", or at least the envelope stands still. A string tied at both ends must have a **node** at each end, that is, a point where the string doesn't move. An **antinode** is a point where maximum displacement occurs. The simplest vibrational mode for a standing wave on a string is when the string vibrates in what is called the fundamental frequency, where the ends are nodes and the middle is an antinode. In more complex waves, the nodes and antinodes may migrate with time. From a node to the nearest antinode is always one quarter of a wavelength, a fact that can be used to help derive relationships between the velocity of the waves on the string, the length, and the frequency of the standing wave.

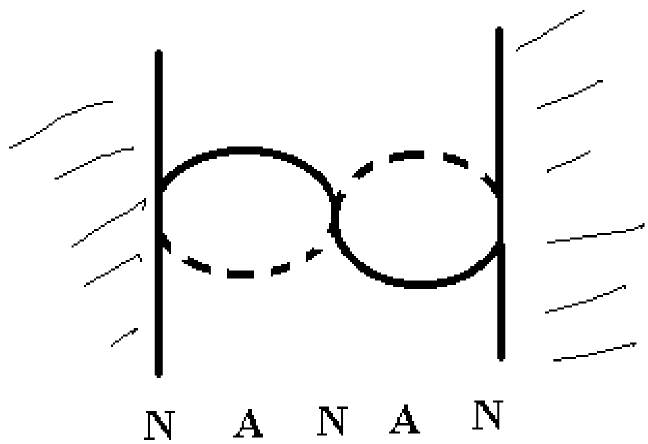
In the figures, there is only one string of length L, but it goes up and down quickly, forming something of an envelope, as shown, but with the previous extreme position indicated by dotted lines. For the fundamental of a string attached at both ends, it is clear that $L = \lambda/2$, which is obtained by counting the number of node-antinode segments along the string. Since $v = f\lambda$, it follows that

$$f_1 = \frac{v}{2L}$$

.

The next simplest situation consists of an antinode in the middle as well as the ends. There are four node-antinode (N-A)segments, each a quarter of a wavelength, so that $\lambda = L$, and

$$f_2 = \frac{v}{L}$$



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In general, adding one more node-antinode pair at a time and continuing the analysis, results in

$$f_n = n\frac{v}{2L} = nf_1 \qquad n = 1,2,3,...$$

where f_n is called the n[th] harmonic.

Example 5. Suppose the third harmonic on a 2 meter string tied at both ends is 780 Hz. (A) What's the fundamental frequency? (B) What is the velocity of waves on the string? (C) What is the density of the string, if the tension is 400 Newtons?

Solution: (A) Since the wave is the third harmonic, n=3 and the fundamental frequency is 780/3=260 Hz. (B) The velocity of a wave can be easily found:

$$f_1 = \frac{v}{2L} \quad \Rightarrow \quad v = 2 f_1 L = 2 \cdot 260 \cdot 2 = 1040 \quad m/s$$

(C) Finally, use the equation for the velocity of a wave on a string to find the linear density:

$$v = \sqrt{\frac{T}{\mu}} \quad \Rightarrow \quad 1040 = \sqrt{\frac{400}{\mu}}$$

Square both sides:

$$1040^2 = \frac{400}{\mu} \quad \Rightarrow \quad \mu = \frac{400}{1040^2} = 3.7 \times 10^{-4} \quad \frac{kg}{m^3}$$

Energy Transport in Strings. The average transmitted power is given by

$$P_{av} = \frac{dE}{dt} = \frac{1}{2} \mu \, \omega^2 A^2 v$$

In this equation, μ is the linear density, ω is the angular frequency, **A** is the amplitude in the y-direction, and **v** the velocity of the wave in the x-direction.

Example 6. A string on a guitar is strung at about 300 Newtons. Suppose such a string masses five grams and is 0.75 meters long, and that when plucked has an amplitude of 2.5 millimeters. (A) What is the speed of the wave? (B) How much power is transmitted along the string, on the average?

Solution: This is just a matter of calculating all the quantities and plugging in. The linear density is given by

$$\mu = \frac{m}{L} = \frac{0.005 \ kg}{0.75 \ meters} = 0.0067 \quad \frac{kg}{meter}$$

The velocity of the waves is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{300}{0.0067}} = 212 \quad m/s$$

The amplitude is given as half a centimeter, or **A = 0.05 m**. The angular frequency can be obtained from the frequency, which can be assumed to be the fundamental frequency of the string. Since the string is attached at both ends:

$$f = \frac{v}{2L} = \frac{212}{2 \cdot 0.75} = 141 \;\; hz$$

So the angular frequency is

$$\omega = 2\pi f = 282\,\pi \;\; rad/\text{sec}$$

Now plug in all these quantities:

$$P_{av} = \frac{1}{2}\mu\,\omega^2 A^2 v = \frac{1}{2} \cdot 0.0067 \cdot 282^2 \cdot \pi^2 \cdot 0.0025^2 \cdot 212 = 3.48 \;\; watts$$

This is enough power to light a small light bulb, say in a pocket flash light.

14.3 Sound Waves.

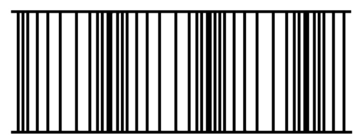
The standard wave equation yields the speed of the wave:

$$v = \sqrt{\frac{B}{\rho}}$$

where ρ is the density of the medium at equilibrium and $\boldsymbol{B}$ is the adiabatic bulk modulus of the matter.

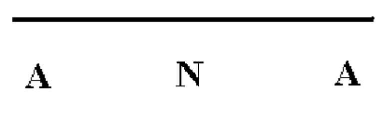
Sound waves are similar to waves on strings, but where string waves are transverse, sound waves are longitudinal. Longitudinal means the oscillations back and forth are in the same direction as the propagation of the wave. This is a little harder to visualize. It's best to imagine a row of ping pong balls, representing the atoms, each separated by a fixed distance. As the wave comes along, it strikes the left most ping pong ball, knocking it over close to the second. Momentarily, therefore, the pressure is increased in the vicinity of the second ping pong ball, because the first ping pong ball is there, too. The collision occurs, knocking the second ball to the right, while the first ping pong ball goes back to the left. Note that where the second ball used to be there is now a gap with no balls, so a reduction in pressure has resulted. This process continues, the gas molecules knocking each other back and forth around equilibrium points. The closed end of a tube always acts as a node, since gas molecules can't be displaced there. In between each pair of nodes there's an antinode, which corresponds to spots where there is maximum displacement of the molecules.

The figure shows another way of imagining this kind of wave. The greater density lines represent a greater pressure (and also gas density). The lesser density lines represents a lesser pressure (and lesser gas density). Notice the alternation of high and low density. This pattern propagates through the air with time, so that at any given point the gas is first high density, then low density, then high density again, and so on. This pattern of high and low density strikes the eardrum and causes it to vibrate, which in turn causes pressure waves inside the ear, which then moves tiny hairs that send signals to the brain.



14.3.1 Standing Waves in Pipes

An analysis of a pipe open at both ends yields the same set of harmonic frequencies result as for a string fixed at both ends:



$$f_n = n \frac{v}{2L}$$

This time, the antinodes are at the ends and the node is in the middle.

Example 7: What length pipe open at both ends will support a fundamental frequency of 600 Hz?

Solution: For the fundamental frequency, n=1. The speed of sound in air, at standard pressure and temperature, is about 345 m/s. We have

$$f = \frac{v}{2L} = 600 \;\; hz \quad \Rightarrow \quad \frac{345}{2L} = 600 \quad \Rightarrow \quad L = \frac{345}{1200} = 0.288 \;\; meters$$

Something like an organ pipe, where one end is open, is somewhat different. The closed end is a node, because air can't move back and forth, there. The other end, where the air is blown, is open, an antinode. Thus $L = \lambda/4$, and $f_1 = v/4L$. Inserting another node-antinode pair and repeating the analysis yields $L = 3\lambda/4$, so that $f_3 = 3v/4L$. It's popular to assert that the even waves are skipped, whereas in actuality they simply aren't there at all. In general, continuing the analysis for a pipe closed at one end, we have

$$f_n = n\frac{v}{4L} = nf_1 \qquad n = 1,3,5...$$

Example 8. Organ pipes can be three or four meters long. What is the fundamental frequency of a pipe four meters long?

Solution: Plug into the equation**.**

$$f = \frac{v}{4L} = \frac{345}{4 \cdot 4} = 21.6 \quad hz$$

14.3.2 Doppler Shift

Sound waves exhibit something called a Doppler shift when the either the source of the sound or the observer is moving. Passing trains and planes make a higher-pitched sound while approaching, shifting abruptly to lower pitch as soon as they pass. When a source moves towards an observer, the wave crests are jammed together, effectively shortening the period ***T*** and thereby increasing the frequency ***f*** , since ***f=1/T***. A source moving away has the opposite effect, making the period of time between crests larger, therefore the frequency smaller. For a moving observer, the effect can be reasoned out similarly. The equation for the effect can be derived from these considerations, and is given by:

$$\frac{f_s}{c \pm v_s} = \frac{f_o}{c \pm v_o}$$

where ***c*** is the velocity of the wave in still air, ***s*** refers to the source, and ***o*** denotes the observer. Notice that each side is identical except for subscripts. Many books offer one or more conventions for choosing the signs, but it's best to use physical reasoning for each of the four possibilities. This is the best way to avoid confusion. Solve the equation for the observer, and then consider the movement of the observer and source independently, choosing the sign in order to get the expected influence on what the observer hears.

Example 9. A train approaches a crossing at 30 m/s, blowing a whistle at 700 Hz at a driver, whose car is trapped between the gates on the track. The driver turns and heads off down the tracks away from the train at 15 m/s. (A) Before he started moving, what frequency did the driver of the car hear? (B) After reaching a speed of 15 m/s, what frequency did the car driver hear?

Solution: Solving the Doppler shift equation for f_o gives

$$f_o = f_s \frac{c \pm v_o}{c \pm v_s}$$

In part (A), $v_o = 0$. The approaching train should jam together the wave crests from the point of view of the observer, hence the period of the waves, $\boldsymbol{T}$, is smaller, which means the frequency, $\boldsymbol{f}$, is bigger, that is, it's a higher pitch. This can be effected only by choosing the minus sign downstairs, so

$$f_o = f_s \frac{c}{c - v_s} = 700 \cdot \frac{345}{345 - 30} = 767 \quad Hz$$

For part (B), everything is as before, except now, while the train approaches the observer, the observer is moving away. This means the time between crests will be increased by observer's movement, reducing the frequency found in part (A). The only way to obtain a reduced frequency as a result of the observer's motion is to choose a negative sign for v_o. This results in

$$f_o = f_s \frac{c - v_o}{c - v_s} = 700 \cdot \frac{345 - 15}{345 - 30} = 733 \ Hz$$

14.3.3 Interference and Beats. Waves can superpose. Mathematically, as seen above, this is a consequence of the superposition principle. Two sound waves superposing result in beats, which correspond to new maxima and minima caused by incomplete destructive interference. the "beat frequency" is given by

$$f_b = f_2 - f_1$$

Example 10. A pair of tuning forks play notes with frequencies ***235 hz*** and ***230 hz*** respectively. What is the beat frequency?

Solution: Dump these into the above equation.

$$f_b = f_2 - f_1 = 235 - 230 = 5 \quad Hz$$

Totally constructive interference between two wave sources of the same frequency occurs when the the path difference is a whole number of wavelengths. Whole number means 0,1,2,3 of course, but not numbers like 2.7. Totally destructive interference occurs when the difference in path length, Δs, is a whole number of wavelengths plus exactly one-half a wavelength. Mathematically:

$$\textit{totally constructive interference}: \quad \Delta s = n\lambda \quad n = \textit{integer}$$
$$\textit{totally destructive interference}: \quad \Delta s = \left(n + \frac{1}{2}\right)\lambda \quad n = \textit{integer}$$

Example 11. Two speakers facing each other and separated by six meters emit sound at 690 Hz. Find all points of constructive interference between the speakers. use c=345 m/s

Solution: These can be readily found. Let the first speaker be a x=0 and the second at x=6, facing each other on the x-axis.

$$\Delta s = (6-x) - x = n\lambda = n\frac{c}{f} = n\cdot\frac{345}{690} = \frac{n}{2}$$

$$6 - 2x = \frac{n}{2} \Rightarrow 3 - x = \frac{n}{4}$$

$$\Rightarrow x = 3 - \frac{n}{4} \qquad n=0,\pm1,\pm2,\ldots\pm6$$

Plugging in all these values of ***n*** will give all the locations of constructive interference.

14.3.4 Intensity. Intensity in sound waves is defined by

$$I = \frac{P}{A}$$

where **P** is the power and **A** is the area over which the power is spread, often taken to be spherically radiated power from a point source, which means $A = \pi r^2$. The intensity level is generally expressed in **dB**, or **decibels**, by the equation

$$\beta = 10\log\left(\frac{I}{I_o}\right)$$

where I_o is a reference intensity taken to be 10^{-12} watts/m^2.

Example 12. Find the intensity level in dB five meters away from a speaker putting out 400 watts of sound power. Assume the wave is spherical.

Solution: The power output is spread over a sphere. Spheres have an area of $4\pi r^2$ where ***r*** is the radius of the sphere. The intensity five meters away is therefore

$$I = \frac{P}{4\pi r^2} = \frac{400}{4\pi 5^2} = 1.27 \ \frac{watts}{m^2}$$

Insert this into the decibel conversion equation:

$$\beta = 10 \log\left(\frac{I}{I_o}\right) = 10 \log\left(\frac{1.27}{10^{-12}}\right) = 121 \quad dB$$

This is very loud, and is the kind of level put out by rock music amplifiers. It damages hearing over a long period of time.

14.4 More Examples

Example 13. Cork dust is placed in a test tube, with a rubber diaphragm across the open end. A sound wave is produced in it. When a musical note of 3,600 Hz is played, the dust assembles into little piles 8 cm apart. What's the velocity of sound in this system?

Solution: This is called a Kundst tube. The cork dust settles into nodes, since in pressure nodes the air is still. Anything falling there will remain there. The pressure antinodes create a sweeping motion, clearing out the dust, which falls into the nodes and tends to stay there. There is one-half a wavelength from one node to the next node, so

$$\frac{\lambda}{2} = 0.08 \; m \qquad \Rightarrow \qquad \lambda = 0.16 \quad meters$$

Since the frequency is known, we can find the speed of propagation of the wave, which is

$$v = f\lambda = 3600 \cdot 0.16 = 576 \; m/s$$

Example 14. Suppose a blind Martian on a motorcycle is speeding head on towards a Mack truck. The truck is traveling at 20 m/s towards the Martian. The driver, seeing the Martian, blows his horn at 500 Hz. The Martian is deaf to frequencies below 700 Hz, but hears the horn and rides safely into a ditch. What minimum speed was the Martian traveling at?

Solution: Both listener and observer are causing the period between wave crests to be shorter, hence frequency greater. Therefore

$$f_o = f_S \frac{c + v_o}{c - v_s}$$

Solve this equation for the speed of the Martian, v_0.

$$f_0 \left(c - v_s \right) = f_s \left(c + v_0 \right)$$

$$v_0 = \frac{f_0}{f_S} \left(c - v_s \right) - c$$

The frequency heard by the Martian has to be at least 700 Hz, or he won't hear it. It could also be greater than that. So plugging in the number 700 gives the minimum speed of the Martian:

$$v_0 = \frac{f_0}{f_S} \left(c - v_s \right) - c = \frac{700}{500} \cdot \left(345 - 20 \right) - 345 = 110 \quad m/s$$

Light waves also exhibit wave behavior and Doppler effects, but this will be taken up in a later chapter.